AN ALGEBRAIC CHARACTERIZATION OF HILBERT LATTICES

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Abstract. In this paper we give an algebraic characterization of the projections lattice of $M_n(\mathbb{C})$ and we extend it to the case of $B(H)$, with H separable Hilbert space.

Contents

1 Introduction

In this paper we give an algebraic characterization of the projections lattice of $M_n(\mathbb{C})$ and we extend it to the case of $B(H)$, with H separable Hilbert space. Such a characterization was founded by several authors following, above all, a topological way (see [So],[Pr],[Dv],[Zi]). The reason of this interest come from the works G.W. Mackey ([Ma]) and Birkhoff-von Neumann ([B-vN]), in which they axiomatize the quantum logic with the lattice of closed subspaces (or equivalently the lattice of projections) of a separable Hilbert space. But we have in mind to use a characterization of Hilbert lattices in order to attach the Connes' embedding conjecture ([Co]) via lattice theory and using a very famous theorem of Kirchberg ([Ki]). For this reason we need a different characterization of Hilbert lattices. It should be based only on algebraic properties and it should describe in the most precise way what happens in the finite dimensional case, i.e. in the case of the $n \times n$ matrices on the complex field. The nearest approach we meet is strangely the first one: during the years 1935-38 J. von Neumann found an axiomatization for the projections lattice of a finite factor, but he used some axioms that we will not use. In particular we don't assume the existence of a transition probability, cutting off at least ten of the eighteen von Neumann's axioms. Obviously our construction loses of generality: it will be valid only in the case of finite factor of type I, i.e. $M_n(\mathbb{C})$. Successively we are able to extend our construction to the separable case. At last we describe a possible second theorem of correspondence that should refine the first one arriving to the "minimal axiomatization" of the projections lattice of $M_n(\mathbb{C})$, in which we are able to cut off other axioms.

So, our approach is similar to the von Neumann's one. This is why we want to compare these two approaches in this preliminary section.

Let us recall the following

Definition 1. (von Neumann, [vN-H2]) A continuous geometry with transition probability is a system $(L, \leq, ^{\perp}, P)$ verifying the following axioms

1. $\leq$ is a partial ordering on L.

2. Each subset of L admits greatest lower bound and least upper bound. We set $l \vee l' = sup\{l, l'\}$, $l \wedge l' = inf\{l, l'\}, 0 = inf(L), 1 = sup(L)$.

3. $\vee$ and $\wedge$ are continuous in the following sense

   (a) If $\{l_i\}$ is an increasing net in L and $l \in L$, then

   $$\bigvee_i (l \wedge l_i) = l \wedge \bigvee_i l_i$$

   (b) If $\{l_i\}$ is a decreasing net in L and $l \in L$, then

   $$\bigwedge_i (l \vee l_i) = l \vee \bigwedge_i l_i$$

4. The modular property holds

   $$l \leq l'' \Rightarrow (l \vee l') \wedge l'' = l \vee (l' \wedge l'') \ \ \forall l' \in L$$

5. $l \to l^\perp$ is an involutory anti-automorphism of L, i.e. $l^{\perp\perp} = l$ and $l \leq l'$ implies $l'^\perp \leq l^\perp$.

6. $l \leq l^\perp$ if and only if $l = 0$.

7. We say that l, l' are inverse if $l \vee l' = 1$ and $l \wedge l' = 0$. This axiom requires that if l, l' and l, l'' are inverse, it cannot be $l' < l''$.

8. $P : L \setminus \{0\} \times L \to [0, 1]$

9. $P(l, l') = 1$ if and only if $l \leq l'$.

10. If $l' \leq l''^\perp$ and $l \neq 0$, then $P(l, l' \vee l'') = P(l, l') + P(l, l'')$.

11. We say that a sequence $\{l_n\}$ is convergent to $l \in L$ if $P(l', l_n) \to P(l', l)$, for all $l' \in L \setminus \{0\}$. This axiom requires that each increasing sequence is convergent.

12. Let $\{l_n\} \subseteq L$. If $P(1, l_n) = constant \neq 0$, $P(l_n, l_m) \to 1$ for $n, m \to \infty$ and $P(l, l_n)$ is convergent for each $l \neq 0$, then $\{l_n\}$ is convergent.

13. For each $\varepsilon > 0$ there exists a $\delta = \delta_\varepsilon > 0$ such that $P(1, l') \leq \delta$ implies $P(1, l \vee l') \leq P(1, l) + \varepsilon$ for all $l \in L$.

14. If $l \in L$ satisfies $l' = (l' \wedge l) \vee (l' \wedge l^{\perp})$ for all $l' \in L$, then $l = 0$ or $l = 1$. This property is called "irreducibility".

15. If T is an $(L, \leq, \perp)$-automorphism, then it is also a P-automorphism, i.e. $P(Tl, Tl') = P(l, l')$, for all $l, l' \in L, l \neq 0$.

16. If $P(1, l) < P(1, l')$ then there exists an $(L, \leq, \perp)$-automorphism T such that $Tl < l'$.

17. If $P(1, l) = P(1, l')$ then there exists an $(L, \leq, \perp)$-automorphism T such that $Tl = l'$ and $Tl'' = l''$ for each $l'' \leq (l \vee l')^{\perp}$.

18. There exist $l_1 < l_2 < l_3 < l_4$.

In [vN-H2] von Neumann proved that a continuous geometry with transition probability is isomorphic to the projection lattice of a finite factor (type I_n or II_1 depends by the values of the dimension function, whose definition we will recall later).
In this paper we obtain the same result in the case I_n, but without using the transition probability. So we are able to cut off the axioms: 8, 9, 10, 11, 12, 13, 15, 16, 17. Moreover we will not use axiom 18 and, apparently, axioms 7 (apparently means that one should study the construction of dimension function deeper in order to see what axioms von Neumann used). On the other hand, we have to add two obvious axioms: L must have the same cardinality of $\mathbb{R}$ and it must contain at least a minimal element. Actually, we will give this last axiom in a (we hope!) more elegant way, generalizing the well-known property of von Neumann algebras: the center of the reduced algebra is the reduced algebra of the center. But we will use this property only to prove that there exist at least one minimal element.

2 Orthocomplemented lattices

In this section we recall the original von Neumann's definition of complete lattice (cfr. [vN-H1]).

Definition 2. A complete lattice is a system $(L, \leq)$ verifying the following axioms:

1. $\leq$ is a partial ordering on L.

2. Each subset of L admits a least upper bound and a greatest lower bound. We set $sup(L) = 1$, $inf(L) = 0$, $inf(l, l') = l \wedge l'$ and $sup(l, l') = l \vee l'$.

3. $\vee$ and $\wedge$ are continuous in the following sense

   (a) If $\{l_i\}$ is an increasing net in L and $l \in L$, then

   $$\bigvee_i (l \wedge l_i) = l \wedge \bigvee_i l_i$$

   (b) If $\{l_i\}$ is a decreasing net in L and $l \in L$, then

   $$\bigwedge_i (l \vee l_i) = l \vee \bigwedge_i l_i$$

4. The modular property holds

$$l \leq l'' \Rightarrow (l \vee l') \wedge l'' = l \vee (l' \wedge l'') \ \ \forall l' \in L$$

Remark 3. More recently one calls lattice only a partially ordered set in which each pair of elements admits sup and inf. In particular a lattice is called modular if it verifies the modular property. We work only on modular lattices (unless a little example in the next section) thus we preferred to follow the original von Neumann's definition. On the other hand, it is just the modular property that make false our characterization in infinite dimension: it is proved in [B-vN] that the projections lattice of $B(H)$, with H infinite-dimensional, is not modular. (see also [Fu]). It might be interesting to see in details what happens without this axiom. We think that our construction can be generalized (maybe generalizing the dimension function), since it is based above all on minimal elements/projections. We will discuss this idea in the last section.

Definition 4. An orthocomplemented lattice is a pair $(L, \perp)$, where L is a complete lattice and $^\perp : L \rightarrow L$ is an involution that satisfies

1. $l \vee l^\perp = 1$

2. $l \wedge l^\perp = 0$

3. $l \leq l'$ implies $l'^\perp \leq l^\perp$

We write also $1 - l$ instead of $l^\perp$.

Note 5. In the definition of continuous lattice with transition probability, von Neumann does not require properties 1. and 2. But one can easily prove that they follow by the von Neumann's sixth axiom and by the following well-known properties.

1. $(l \vee l')^{\perp} = l^{\perp} \wedge l'^{\perp}$
2. $(l \wedge l')^{\perp} = l^{\perp} \vee l'^{\perp}$

Let L be an orthocomplemented lattice.

Definition 6. Let $l \in L$. We set $\perp(l) = \{l' \in L : l' \leq 1 - l\}$. Elements belonging into $\perp(l)$ are said to be orthogonal to l.

Definition 7. Let $l, l' \in L$. We said that l commutes with l' if

$$l = (l \wedge l') \vee (l \wedge l'^{\perp})$$

$c(l)$ will denotes the set of elements which commute with l.

It is well-known that

1. $\{0, 1, l\} \subseteq c(l)$
2. $l' \in c(l) \Leftrightarrow l \in c(l')$

Thus the relation of commutation is symmetric and we can say that "l,l' commute".

Definition 8. We set

$$C(L) = \bigcap_{l \in L} c(l)$$

Elements belonging into $C(L)$ are called central. Certainly $C(L) \supseteq \{0, 1\}$. L is called factorial if $C(L) = \{0, 1\}$, abelian if $C(L) = L$.

Note 9. Factoriality corresponds to irreducibility (14th von Neumann's axiom).

Lemma 10. *Let $l \in L$. Then $\perp(l) \subseteq c(l)$*

Proof. Let $l' \leq 1 - l$ and thus $l \leq 1 - l'$. We have

$$[l \wedge (1 - l')] \vee (l \wedge l') = l \vee 0 = l$$

thus $l' \in c(l)$. $\square$

Proposition 11. *Let* $l_3 \in c(l_1) \cap c(l_2)$*, then*

$$(l_1 \vee l_2) \wedge l_3 = (l_1 \wedge l_3) \vee (l_2 \wedge l_3)$$

Proof. Applying the modular law (sometimes!), one has

$$(l_1 \vee l_2) \wedge l_3 = [l_1 \vee (l_2 \wedge l_3) \vee (l_2 \wedge l_3^{\perp})] \wedge l_3 =$$

$$= (l_2 \wedge l_3) \vee [(l_1 \vee (l_2 \wedge l_3^{\perp})) \wedge l_3] =$$

$$= (l_2 \wedge l_3) \vee [((l_1 \wedge l_3) \vee (l_1 \wedge l_3^{\perp}) \vee (l_2 \wedge l_3^{\perp})) \wedge l_3] =$$

$$= (l_2 \wedge l_3) \vee [(l_1 \wedge l_3) \vee (((l_1 \wedge l_3^{\perp}) \vee (l_2 \wedge l_3^{\perp})) \wedge l_3)] =$$

$$= (l_2 \wedge l_3) \vee (l_1 \wedge l_3)$$

□

Corollary 12. *An othocomplemented lattice* L *is abelian if and only if for each* $l, l', l'' \in L$ *the following holds*

$$(l \vee l') \wedge l'' = (l \wedge l'') \vee (l' \wedge l'')$$

Thus abelian lattices coincide with those are classically called distributive lattices or Boole algebras.

Proof. If L is abelian, one can apply Prop. 11. Conversely, one can apply the formula with $l' = 1 - l$. □

Proposition 13. *Let* $l, l' \in \perp (l)$*. If* $l \vee l' = l \vee l''$*, then* $l' = l''$*.*

Proof. By Lemma 10 we have $l \in c(l') \cap c(l'')$. Thus, using Prop. 11,

$$l'' = (l \wedge l'') \vee (l'' \wedge l'') = (l \vee l'') \wedge l'' =$$

$$= (l \vee l') \wedge l'' = (l \wedge l'') \vee (l' \wedge l'') = l' \wedge l''$$

Whence $l'' = l' \wedge l''$ and consequently $l'' \leq l'$. Changing l' and l'' we can find the reverse inequality. □

Remark 14. Let $l \in L$, we consider the complete lattice $L' = L \wedge l = \{l' \wedge l, l' \in L\}$. This can be orthocomplemented in a natural way setting (for each $l' \leq l$)

$$l - l' = (1 - l') \wedge l$$

Notice that one have to use the modular property to prove that $l' \vee (l - l') = l$. Indeed

$$l' \vee (l - l') = l' \vee [(1 - l') \wedge l] = [l' \vee (1 - l')] \wedge l = 1 \wedge l = l$$

Definition 15. An element $l \in L$ is called abelian if $L \wedge l$ is an abelian lattice. Let $Ab(L)$ be the set of abelian element in L.

Remark 16. Obviously a Boole algebra can be characterized as a lattice for which $Ab(L) = L$.

Definition 17. A factorial lattice is said "of type I" if it admits at least a non-zero abelian element.

Remark 18. By definition of orthocomplement in $L \wedge l$, it follows that $C(L) \wedge l \subseteq C(L \wedge l)$. The reverse inclusion holds for Boole algebras and it could hold for other important classes of orthocomplemented lattices.

Definition 19. An R-lattice is an orthocomplemented lattice in which the following property of restriction of the central elements holds

$$C(L \wedge l) = C(L) \wedge l \quad \forall l \in L$$

We repeat that this property generalized the well-known property of von Neumann algebras: the center of the reduced algebra is the reduced algebra of the center.

Problem 20. Do there exist orthocomplemented lattices which are not R-lattices? also among the factorial ones?

We conclude this preliminary section giving a nice characterization of the orthogonal complement. We recall the following

Definition 21. Let $l \in L$. An inverse of l is an element $l' \in L$ such that $l' \vee l = 1$ e $l \wedge l = 0$. $inv(l)$ will denote the set of the inverse of l.

Proposition 22. *Let $l \in L$. One has*

$$c(l) \cap inv(l) = \{1 - l\}$$

Proof. Certainly $1 - l \in c(l) \cap inv(l)$. Now let $x \in c(l) \cap inv(l)$, then

$$x = (x \wedge l) \vee (x \wedge (1 - l)) = x \wedge (1 - l)$$

thus $x \leq 1 - l$. Now we assume that $x \neq 1 - l$, then $1 - l - x$ is defined and not zero. Now we observe that if l, l' are two elements in L and $l' \leq l$, we defined $l - l' = l \wedge (1 - l')$. Now we have $1 - l' \leq l$ and thus

$$l - (1 - l') = l \wedge (1 - (1 - l')) = l \wedge l'$$

Applying this observation with $l = 1 - l$ and $l' = 1 - x$ we have

$$1 - l \vee x = (1 - l) \wedge (1 - x) = 1 - l - x \neq 0$$

ant thus the absurd: $l \vee x \neq 1$. $\square$

Definition 23. $l, l' \in L$ are said to be independent if $l \wedge l' = 0$.

Remark 24. To prove the implication $x \in c(l) \cap inv(l) \Rightarrow x \leq 1 - l$ we used only the independence between l and l'. Thus we can interpret $1 - l$ as the sup of elements which are independents and commuting with l .

3 Some examples

In this section we give some examples of othocomplemented lattices.

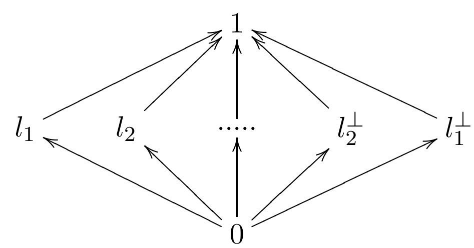
1. For each set X, the set of its subsets $P(X)$ is a Boolean algebra. It is well-known (see [Bi]) that each finite Boolean algebra can be obtained in this way. The Stone representation theorem (see [St] or [Jo]) give the following deep characterization: each Boolean algebra is isomorphic to the set of clopen subsets (subsets which are open and closed) of an Hausdorff extremely disconnected topological space.

2. The projections lattice of a von Neumann algebra is always an infinite orthocomplemented lattice. It is abelian (resp. factorial) if and only if the von Neumann algebra is abelian (resp. a factor).

3. Let $n \in \mathbb{N}$. If n is odd, there are no orthocomplemented lattices with n elements. On the other hand, if n is even, and equal to $2m$, it is possible to construct a factorial lattice with n elements. Indeed we set

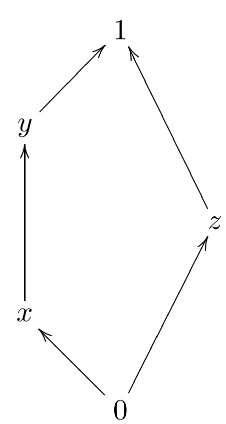
$$L_m = \{0, l_1, 1 - l_1, l_2, 1 - l_2, ...l_m, 1 - l_m\}$$

with the conditions $l \wedge l' = 0, l \vee l' = 1$ for each $l \neq l'$. This lattice can be represented in the following way



Chevalier showed that with these lattices, together with the distributive ones, one can construct each finite orthocomplemented lattice. Indeed one can define (in a obvious way) the algebraic product of a finite family of orthocomplemented lattices with the lexicographic order and setting $(l_1, ...l_n)^\perp = (l_1^\perp, ...l_n^\perp)$. Chevalier proved that each finite orthocomplemented lattice is product of $\{0, 1\}^n$ with a finite family of type L_m lattices. (see also [Sv]).

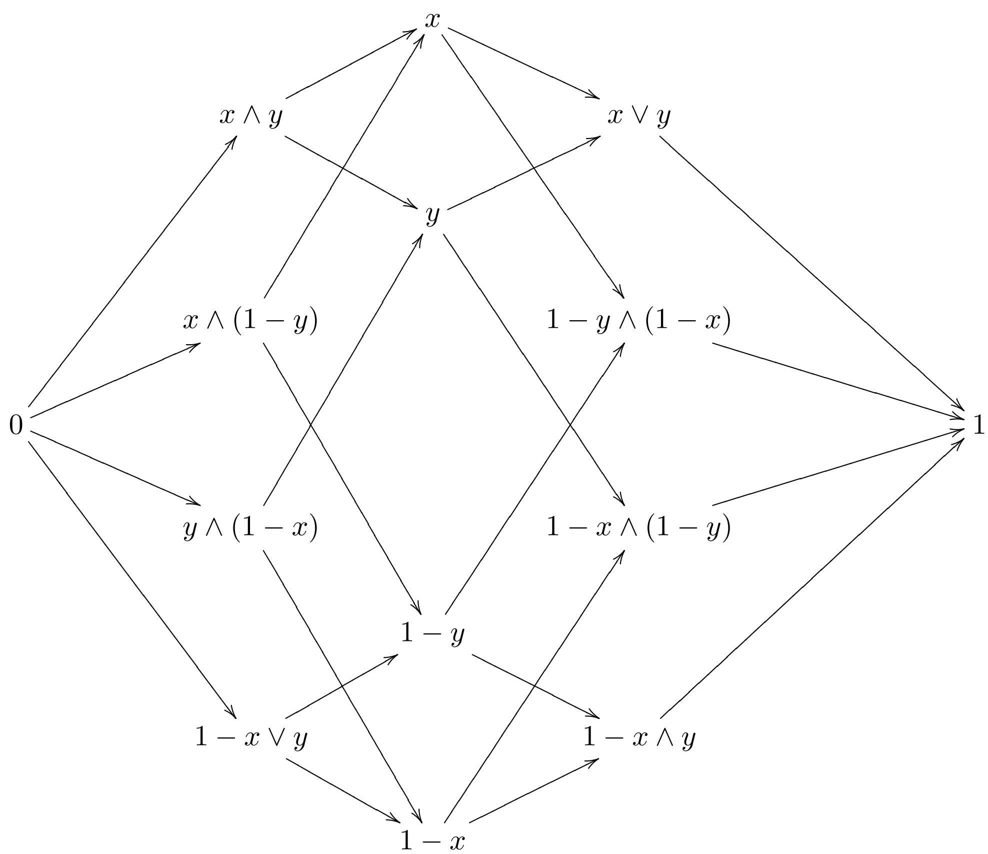
4. More recently lattices which are not modular are studied. The simplest of those is the pentagon N_5



It does not satisfies the modular property. Indeed

$$(x \vee z) \wedge y = y \neq x = x \vee (z \wedge y)$$

In [G] it is proved that a lattice verifies the modular property if and only if it does not contain sublattices isomorphic to N_5.



Notice that N_5 is not an orthocomplemented lattice. To construct an orthocomplemented lattice which does not verify the modular property one can consider the previous one. Indeed it is clearly an orthocomplemented lattice, in the sense that it satisfies the first three property of the definition of lattice and all those of the definition of the orthocomplementation. But it is not modular (you find with $\{x, 1-x, x \vee y\}$). Moreover, a direct calculation shows that $C(L) = L$. On the other hand it is not distributive (otherwise it should be modular!). Notice that, in order to obtain the equivalence in Cor.12, we used the modular property to prove Prop. 11. More generally the following proposition holds

Proposition 25. *Let L be an orthocomplemented non-modular lattice. One considers the following three properties*

(a) L is modular.

(b) L is abelian.

(c) L is distributive.

Then $c) \Leftrightarrow a) \wedge b)$. *That is: the third property holds if and only if both the other two hold.*

Problem 26. What is the smallest example of orthocomplemented lattice which is neither abelian nor factorial?

Notice that the previous one is not the smallest example of orthocomplemented lattice which is abelian without being distributive. The hexagon is a much more trivial example!

4 Equivalence relations on orthocomplemented lattices

In [vN-H1] von Neumann showed that a powerful mean to know the structure of an orthocomplemented lattice is given by its equivalence relations. In the next three sections we want to run again the street opened by von Neumann, abstracting some concepts.

Definition 27. Let L be a lattice and $\sim$ an equivalence relation on L. We say that $l \in L$ is $\sim$dominated by $l' \in L$ (and we write $l \leq_{\sim} l'$) if and only if there exists $l'' \leq l'$ such that $l'' \sim l'$. We write $l <_{\sim} l'$ if $l'' < l'$.

Obviously one is interested to equivalence relation that are compatible with the ordering in some sense.

Definition 28. An equivalence relation on an orthocomplemented lattice is called regular if

1. $l \sim 0 \Leftrightarrow l = 0$
2. $l \geq l'$ and $l \leq_{\sim} l'$ imply $l \sim l'$ (order compatibility)
3. For each $l, l' \in L$ one and only one of the followings hold $l \sim l'$, $l \leq_{\sim} l'$, $l' \leq_{\sim} l$.

4. If $\{l_i\}$ and $\{l'_i\}$ are two families of mutually orthogonal elements and they are such that $l_i \sim l'_i$ for each i, then
$$\bigvee_i l_i \sim \bigvee_i l'_i$$
5. Conditions $l' \leq l, l' \sim l$ imply $l' = l$ (finiteness property).

Definition 29. A complete lattice is called irreducible if only 0 and 1 have unique inverse (see Def.21)

Remark 30. We recall that irreducibility corresponds (in orthocomplemented lattices) to factoriality (see. [vN-H2], pg. 14).

We can join some von Neumann's results to obtain the following

Theorem 31. (von Neumann) *Let L be a complete irreducible lattice. Setting $l \sim l'$ if and only if they are a common inverse (see Def.21), then $\sim$ is a regular equivalence relation on L.*

Proof. Properties 2,3,4 e 5 are already proved by von Neumann in [vN-H1]. The first one is clear: if $l \sim 0$, then $l, 0$ have a common inverse. But the unique inverse of 0 is 1 and thus $l = 1 \wedge l = 0$. $\square$

Definition 32. We say that two elements are perspective if they are a common inverse.

Thus each irreducible complete lattice (and in particular each orthocomplemented factorial lattice) admits a regular equivalence relation. In this case one can ask if this relation is also compatible with the orthogonality in some sense. Property 4. is already a compatibility with the orthogonality, but we will prove something more: there exists substantially only one regular relation (which coincide with the perspective relation, i.e. with the Murray-von Neumann relation on projections) and thus it verifies the parallelogram law: $l \vee l' - l \sim l' - l \wedge l'$.

Proposition 33. *Let $\sim$ be a regular equivalence relation on L. If $l \sim l'$ and $l_1 \leq l$, then $l_1 \leq_\sim l'$.*

Proof. By definition one has $l_1 \leq_\sim l'$ or $l' \leq_\sim l_1$. We assume the last one and we prove that actually the first one holds in the form: $l_1 \sim l'$. Indeed $l \sim l' \leq_\sim l_1$ and thus, by transitivity, $l \leq_\sim l_1$. On the other hand, by hypothesis, $l_1 \leq l$ and thus, by order compatibility, it follows that $l_1 \sim l$ and thus $l_1 \sim l'$. $\square$

5 Minimal elements in factorial lattices

Let L be a factorial R-lattice and $\sim$ be a regular equivalence relation on L. We recall the property R: $C(L) \wedge l = C(L \wedge l)$ for each $l \in L$. We don't know if there exists factorial lattices which do not verify it.

Definition 34. A non-zero element $l \in L$ is called minimal if for each $l' \in L$ one and only one of the following holds: $l \wedge l' = 0$, or $l \wedge l' = l$. Let $Min(L)$ denote the set of minimal elements of L and with $Min(l)$ the set of minimal elements of L which are less then l.

Note 35. In literature minimal elements are also called atoms. We prefer to call them minimal for keeping a sense of continuously with the theory of Operator Algebras.

Corollary 36. *If l is minimal and $l' \sim l$, than also l' is minimal.*

Proof. Let $l'_1 \leq l'$ and thus (by Prop.33) $l'_1 \leq_\sim l$ and consequently $l'_1 \sim 0$, or $l'_1 \sim l$. The properties of $\sim$ then guarantee $l'_1 = 0$, or, by finiteness, $l'_1 = l'$. $\square$

Here is one of the main result to develop our approach.

Proposition 37. *Let L be a type I factorial lattice. Each element of L contains a minimal element.*

Proof. Let $l \neq 0$, $l \in Ab(L)$. Factoriality and property R imply $\{0, l\} = C(L) \wedge l = C(L \wedge l)$; and thus $L \wedge l$ is still factorial. Since it is also abelian, it must be $L \wedge l = \{0, l\}$. Thus l is minimal. Now let $l' \in L$. It must be $l' \leq_\sim l$, or $l \leq_\sim l'$. In the first case $l' \sim l_1 \leq l$. Thus $l' \sim 0$ (and therefore $l' = 0$), or $l' \sim l$ (and therefore l' is minimal, by Cor.36). In the second one l' contains an element equivalent to l. This element must be minimal, being l minimal (still by Cor.36). $\square$

Corollary 38. *Let L be a type I factorial lattice and $\{l_i\}$ a family of elements which are maximal in the properties to be minimal and mutually orthogonal. Then $\bigvee_i l_i = 1$.*

Proof. If $l = 1 - \bigvee_i l_i \neq 0$, we can find (see Prop.37) a minimal element $l_1 \leq l$, contradicting maximality. □

Corollary 39. *Let L be a type I factorial lattice. Each element of L is sup of a family of minimal and mutually orthogonal elements.*

6 The dimension function

In this section we give a sketch of the construction of the dimension function in an irreducible complete lattice.

Definition 40. A system $\{l_i\}_{i \in I}$ of elements in L is called independent if for each partition $\{J, K\}$ of I one has

$$\bigvee_{i \in J} l_i \wedge \bigvee_{i \in K} l_i = 0$$

Note 41. In this section one can change the word "independent" with "orthogonal". Th.31 is still valid.

Definition 42. Let L be a complete lattice and $\sim$ a regular equivalence relation on L. A dimension function $\sim$-compatible is a map $D : L \to [0, 1]$ such that

1. $D(0) = 0, D(1) = 1$
2. $D(l \vee l') + D(l \wedge l') = D(l) + D(l')$
3. $D(l) = D(l') \Leftrightarrow l \sim l'$
4. $D(l) \leq D(l') \Leftrightarrow l \leq_\sim l'$
5. If $\{l_i\}$ is a finite or countable independent system, then

$$D(\bigvee l_i) = \sum D(l_i)$$

Von Neumann proved in [vN-H1] that choosing $\sim$ as the perspective equivalence relation, then there exists a unique dimension function $\sim$-compatible. Moreover the image $D(L)$ can be only one among the following sets

1. $\Delta_n = \{0, \frac{1}{n}, \frac{2}{n}, ... \frac{n-1}{n}, 1\}$
2. $\Delta_\infty = [0, 1]$

Here we give a sketch of the construction of the dimension function. The complete construction can be founded in the first five chapters of [vN-H1]. Let us precise that this construction also works in our case (factorial R-lattices), since factoriality and irreducibility coincide for orthocomplemented lattices.

(I step)
Since we want define a class function (i.e. constant on each equivalence class) which describes the order of the equivalence class through the order of their values, the first step consists of defining some operations between equivalence classes. More precisely, we set $\mathcal{L} = L/\sim$ and we denote $A, B, C...$ the elements of $\mathcal{L}$. Von Neumann himself proved that if there exist $a \in A, b \in B$ such that $a \wedge b = 0$ then one can well-define a unique class $A \vee B$ (that is the class containing $a \vee b$). Analogously , if there exist $a \in A, b \in B$ such that $a \geq b$, one can well-define a unique class $A - B$ (that is the class containing $a - b$).

(II step)
After proving some algebraic properties of the operations between classes, von Neumann gave the following crucial

Definition 43. Let A_0 be the class containing $0 \in L$ and $A \in \mathcal{L}$. We set $0A = A_0$. Now, supposing defined $(n-1)A$ and that there exists $(n-1)A \vee A$, than we define $nA = (n-1)A \vee A$. Otherwise nA is not defined.

In order to understand the idea of this definition, it is better to think at factorial R-lattices of type I. In this case we know that each element is sup of a family of minimal elements. Now we assume the following fact (that we will prove later)

Theorem 44. *Let L be a factorial R-lattice of type I and $\{l_i\}_{i\in I}, \{l'_j\}_{j\in J}$ two families of mutually orthogonal and minimal elements. Then*

1. $|I|$ and $|J|$ are finite numbers.

2. $\bigvee_i l_i \sim \bigvee_j l'_j$ if and only if $|I| = |J|$.

Let $A \in \mathcal{L}$. Thanks to this theorem we can define m_1 as the number of minimal elements appearing in some decomposition of $1 \in L$ and m_A as the number of minimal elements appearing in some decomposition of $a \in A$. Let A_{min} be the class containing each minimal element of L and let n_A be the greatest integer for which nA is defined. In this case von Neumann's idea is to determine m_1 as $n_{A_{min}}$ (indeed $nA_{min} \vee A_{min}$ is defined until one takes a maximal family of minimal and mutually orthogonal elements), then to determine $n_A = n_{A_{min}} - m_A$ and lastly to define the dimension of the class A as $m_A/n_{A_{min}}$. One can follow this idea also in more general cases: the key is to understand what means, given two classes A and B: n is the greatest integer for which nA is into B.

Definition 45. Let $A, B \in \mathcal{L}$. We set $A < B$ if and only if there exist $a \in A, b \in B$ such that $a <_\sim b$.

Now we can enunciate one of the most important theorem of this theory.

Theorem 46. *Let $A_0 \neq A, B \in \mathcal{L}$. There exist a unique pair (n, B_1), where $n \geq 0$ is an integer and $B_1 < A$ is a class such that*

$$B = nA + B_1$$

We set $[B : A] = n$.

Now the construction would be concluded in the case of factorial lattices of type I. In the general case in which we have not minimal elements, we need a further step.

(III step)

Definition 47. A class $A_0 \neq A \in \mathcal{L}$ is said to be minimal if there no exists $B \neq A_0$ such that $B < A$.

The case in which A is not minimal is optimally described by von Neumann with the following

Theorem 48. *If A is not minimal, then there exist $B \neq A_0$ such that $2B \leq A$.*

Definition 49. A minimal sequence $\{A_n\}$ of elements $\neq A_0$ is one containing but one element B which is minimal, or containing a denumerable infinitude of elements such that $2A_{n+1} \leq A_n$.

Using th.48 we have the existence of minimal sequence. The following theorem allows to define the dimension function.

Theorem 50. *Let $A_0 \neq A \in \mathcal{L}$. Then the following limit exists and it is finite and positive*

$$(A : A_1) = lim_{i\to\infty} \frac{[A : A_i]}{[A_1 : A_i]}$$

where A_1 denotes the class in which $1 \in L$ belong. (If $\{A_i\}$ consists of one minimal element B, we mean by $lim_{i\to\infty}$ the value at $A_i = B$.).

(IV step)

Definition 51. Let $l \in L$ and A_l be the class containing l. We set $D(l) = (A_l : A_1)$ if $l \neq 0$. Otherwise $D(0) = 0$.

The dimension function verifies the following properties.

1. $D(l) \in [0,1]$ for each $l \in L$. $D(0) = 0, D(1) = 1$
2. $D(l \vee l') + D(l \wedge l') = D(l) + D(l')$
3. $D(l) = D(l') \Leftrightarrow l \sim l'$
4. $D(l) \leq D(l') \Leftrightarrow l \leq_\sim l'$
5. if $\{l_i\}$ è is a finite or countable independent system, then

$$D(\bigvee l_i) = \sum D(l_i)$$

6. $D(L)$ è is one of the following sets
   (a) $\Delta_n = \{0, \frac{1}{n}, \frac{2}{n}, ... \frac{n-1}{n}, 1\}$
   (b) $\Delta_\infty = [0,1]$

We conclude this section proving some easy facts descending by von Neumann construction. Let, for the rest of the section, L be a factorial R-lattice. At first we give the following

Definition 52. L is said of type I_n if it admits a dimension function D such that $D(L) = \Delta_n$. It is said to be of type II_1 if it admits a dimension function D such that $D(L) = \Delta_\infty$.

Remark 53. An easy consequence of the correspondence theorem (see th.70) is that the notion of type do not depend by the choice of the $\sim$dimensionable equivalence relation.

Remark 54. Notation I_n agree with the locution "of type I". Indeed, by the following prop.56 it follows that a factorial R-lattice is of type I_n (for some n) if and only if it is of type I.

Definition 55. A reference on L is given by the choice of a maximal family with respect to the properties: mutually orthogonal and minimal.

Proposition 56. *Let L of type I_n. Then*

1. *$l \in Min(L)$ if and only if $D(l) = \frac{1}{n}$*
2. *if R is a reference on L, then $|R| = n$.*

Proof.
1. Let $l \in Min(l)$. If it were $D(l) = 0$, it would be $l \sim 0$ and thus $l = 0$. Instead, if it were $D(l) = m/n$, with $m > 1$, one could consider $l' \in L$ such that $D(l') = 1/n$. Whence $0 \neq l' \leq_\sim l$ and thus l could not be minimal. Conversely it is clear using the finiteness property of minimal elements.
2. We suppose for example $|R| = m > n$ (the other case is just the same). Let $l_1, ...l_m \in R$. We obtain the following absurd

$$1 = D(1) \geq D(l_1 \vee ... \vee l_m) = \sum_{i=1}^{m} D(l_i) = \sum_{i=1}^{m} 1/n = m/n > 1$$

□

Now we can prove th.44.

Proof. 1. Using Prop.56, we have $|I|, |J| \leq n$.

2. If $|I| = |J|$ then $l = \bigvee_i l_i \sim \bigvee_j l'_j = l'$ since $\sim$ is regular. Conversely, we assume for example that $|I| \leq |J|$, then there exists $J' \subseteq J$ such that $|I| = |J'|$ and thus, by the regularity of $\sim$, we have $\bigvee_{i \in I} l_i \sim \bigvee_{j \in J'} l'_j = l'_1$. Whence $l'_1 \leq l'$ and $l'_1 \sim l'$. By the finiteness property it follows that $l'_1 = l'$ and thus, since l'_j are mutually orthogonal, $|J| = |J'|$.

□

7 Factorial W^*-lattices of type I_n

Since we want axiomatize the projection lattice of $M_n(\mathbb{C})$, we are interested in the case in which the lattice contains infinite non-countable elements. This property does not hold in generale, since one can easily construct finite factorial lattices for example of type I_2: $L = \{0, x, 1-x, y, 1-y, 1\}$ in which $x, 1-x, 1-y, y$ have dimension $1/2$. Thus they are minimals and consequently x (resp. y) commutes only with $0, x, 1-x, 1$ (resp. $0, y, 1-y, 1$). Whence L is factorial.

Definition 57. A factorial W^*-lattice of type I_n is a factorial R-lattice of type I_n that contains infinite non-countable elements.

8 Geometry of minimal elements

Let L be a factorial W^*-lattice of type I_n. We start proving that also $Min(L)$ has the same cardinality of L.

Proposition 58.

$$|Min(L)| = |\mathbb{R}|$$

Proof. It follows by Cor.39 and Th.44 that each element of L is sup of a finite family of minimal elements. Thus L has the same cardinality of the set of the finite subsets of $Min(L)$, which has the same cardinality of $Min(L)$. □

Definition 59. Let A be a set, $a_0 \in A$ and $A' \subseteq A \times A$. We say that A' is a_0-right-separated if $(A \times \{a_0\}) \cap A' = \emptyset$.

Lemma 60. *Let A, B be two equipotent sets, $A' \subseteq A$, $B' \subseteq B$ be also equipotent, $A'' \subseteq A \times A$ a_0-right-separated and equipotent to the b_0-right-separated set $B'' \subseteq B \times B$. Then there exists a bijection $\Psi : A \to B$ such that*

1. $\Psi|_{A'} : A' \to B'$ *is a bijection*

2. $(a_1, a_2) \in A'' \Leftrightarrow (\Psi(a_1), \Psi(a_2)) \in B''$

Proof. Let us consider the following subsets of $A \times A$: $A_1 = A' \times \{a_0\}, A_2 = (A \backslash A') \times \{a_0\}, A_3 = A'', A_4 = (A \times A) \backslash (A'' \cup A \times \{a_0\})$. They are a partition of $A \times A$ and they are respectively equipotent to $B_1 = B' \times \{b_0\}, B_2 = (B \backslash B') \times \{b_0\}, B_3 = B'', B_4 = (B \times B) \backslash (B'' \cup B \times \{b_0\})$, which is a partition of $B \times B$. Thus there exist bijections $\psi_1 : A_1 \to B_1$, $\psi_2 : A_2 \to B_2$, $\psi_3 : A_3 \to B_3$ and $\psi_4 : A_4 \to B_4$. Now we can join all these maps to obtain a bijection $\Psi_1 : A \times A \to B \times B$, which, restricted to $A \times \{a_0\}$, gives a bijection between it (and thus between A) and $B \times \{b_0\}$ (and thus with B) that satisfies the required properties. $\square$

It follows an unexpected and decisive result.

Definition 61. Let I be a set and $l^2(I)$ the standard Hilbert space on I. Let $l^2(I)_1$ be the unit sphere of $l^2(I)$, that is

$$l^2(I)_1 = \{x \in l^2(I) : ||x|| = 1\}$$

We define the following equivalence relation on $l^2(I)_1$: $xEy \Leftrightarrow \exists \theta \in [0, 2\pi) : x = e^{i\theta} y$, in which transitivity follows by summing θ modulo 2π.

Theorem 62. *Let L be a factorial W^*-lattice of type I_n on which a reference $R = \{r_1, ...r_n\}$ is fixed. Then there exists a bijection $\Psi : Min(L) \to l^2(R)_1/E$ such that*

1. $\Psi(r_i)$ *is the i-th element of the canonical orthonormal basis of the Hilbert space* $l^2(R)$.

2. $m \perp n \Leftrightarrow \Psi(m) \perp \Psi(n)$ *in the Hilbert space* $l^2(R)$.

Proof. Let $A = Min(L) \cup \{1\} \subseteq L, A' = R, A'' = \{(m,n) \in A \times A : m \perp n\}, B = l^2(R)_1/E \cup \{1\} \subseteq B(l^2(R)), B' =$"canonical orthonormal basis of $l^2(R)$", $B'' = \{(x,y) \in B \times B : x \perp y\}$. It is enough to prove that those sets satisfy the hypothesis of lemma 60 and that $\Psi(1) = 1$. In this case we can restrict Ψ and obtain a bijection $\Psi : A \to B$ without modifying its properties. Certainly A'' and B'' are 1-right-separated. Now, since R is finite, then $l^2(R)_1$ has the same cardinality of $\mathbb{R}$, that is the same cardinality of A (by Prop.58). Moreover A' is equipotent to B' since each basis of $l^2(R)$ has the same cardinality of R and thus the same cardinality of A'. Lastly A'' is equipotent to A and B'' is equipotent to B and thus A is equipotent to B. It remains to prove that $\Psi(1) = 1$: it is sufficient to observe that in the proof of lemma 60 one can always choose ψ_1 such that $\psi_1(a_0, a_0) = (b_0, b_0)$. □

After this fundamental result we prove some easy lemmas on the behavior of the minimal elements. Also these lemmas will be crucial in the proof of the correspondence theorem.

Lemma 63. *For each $l \in L$, one has $\bigvee_{m \in Min(l)} m = l$.*

Proof. Since $m \leq l$ for each $m \in Min(l)$, one certainly has $\bigvee_{m \in Min(l)} m \leq l$. Conversely, if $l - \bigvee_{m \in Min(l)} m \neq 0$, then there exists $m' \in Min(l - \bigvee_{m \in Min(l)} m)$. This is a contradiction, because we have already get all the minimal elements dominated by l. □

Lemma 64. *Let $l, l' \in L$. Then $l \leq l'$ if and only if $Min(l) \subseteq Min(l')$.*

Proof. If $l \leq l'$ and $m \in Min(l)$ one certainly has $m \in Min(l')$. Conversely, using lemma 63, one has

$$l = \bigvee_{m \in Min(l)} m \leq \bigvee_{m \in Min(l')} m = l'$$

□

Lemma 65. *Let $l, l' \in L$. Then $\perp (l \vee l') \subseteq \perp (l) \cap \perp (l')$.*

Proof. Let $x \in \perp (l \vee l')$, then, by definition, $x \leq 1 - (l \vee l')$. Now, we know that $l, l' \leq l \vee l'$ and thus $1 - l \vee l' \leq 1 - l, 1 - l'$. So $x \leq 1 - l, 1 - l'$ and $x \in \perp (l) \cap \perp (l')$. □

Lemma 66. *Let $l, l' \in L$. Then*

$$Min(l \vee l') \cap \perp (l) \cap \perp (l') = \emptyset$$

Proof. Let $x \in \perp (l) \cap \perp (l')$, then $x \leq 1 - l, 1 - l'$. Consequently $x \leq (1 - l) \wedge (1 - l') = 1 - (l \vee l')$. So $x \in \perp (l \vee l')$ and thus it can not be dominated by $l \vee l'$. $\square$

Lemma 67. *Let $m, n \in Min(L)$. Then $m \sim n$.*

Proof. It must be $m \leq_{\sim} n$ or $n \leq_{\sim} m$. Since each non zero element is not equivalent to 0 and since m, n are minimal, the it must be exactly $m \sim n$. $\square$

Lemma 68. *For each $l' \leq l$, one has $Min(l - l') = Min(l) \cap \perp (l')$.*

Proof. If $x \in Min(l - l')$ one certainly has $x \in Min(l)$. Moreover $x \leq l - l' \leq 1 - l'$ and thus $x \in \perp (l')$. Conversely, if $x \in Min(l) \cap \perp (l')$, then $x \leq 1 - l'$ is minimal. But $x \leq l$ and thus $x \leq (1 - l') \wedge l = l - l'$ is minimal. $\square$

9 The correspondence theorem

In this section we finally prove that the concept of factorial W^*-lattice of type I_n axiomatizes the projection lattice of $M_n(\mathbb{C})$.

Notation 69. P_n will denote the projection lattice of $B(M_n(\mathbb{C}))$.

Theorem 70. *The map $M_n(\mathbb{C}) \to P_n$ sends type I_n factors in factorial W^*-lattices of type I_n. Conversely, if L is a factorial W^*-lattice of type I_n, then $L \cong P_n$.*

Proof. Of course P_n is a factorial W^*-lattice of type I_n, using the normalized trace as dimension. Conversely, let $\{l_i\}_{i=1}^n$ be a reference on L and we set $H = l^2(I)$. We map l_i in the projection e_{l_i} of H onto the i-th addend of the direct sum $H = \mathbb{C}^n$. Now, if $l \in L$ is minimal, we map it in the projection of H onto $\Psi(l)\mathbb{C}$. Now, if $l \in L$, we can write $l = \bigvee l_j$ (where the l_j are minimal and mutually orthogonal). Since all the $\Psi(l_j)$ are still minimal and mutually orthogonal in $M_n(\mathbb{C})$, we can map l in the projection e_l of H onto $\bigoplus_{j \in J} \Psi(l_j)\mathbb{C}$. In this way, we have constructed a map from L into P_n. Now, if $e \in P_n$,

we write eH as direct sum of one-dimensional subspaces and then observe that each one-dimensional subspace is of the form $a\mathbb{C}$ (for a unique $a \in l^2(I)_1/E$). So, recalling that Ψ is a bijection, we have that eH comes from $\bigvee \Psi^{-1}(a)$, where a runs over a decomposition of eH. Thus we have a bijection between L and P_n. Now we have to prove that they are the same lattice structure. Let $\Psi : L \to P(B(H))$ be the bijection we have already constructed. We have to prove the following properties:

1. Ψ preserves the lattice structure. That is
   (a) $\Psi(l) \leq \Psi(l') \Leftrightarrow l \leq l'$
   (b) $\Psi(l \wedge l') = \Psi(l) \wedge \Psi(l')$
   (c) $\Psi(l \vee l') = \Psi(l) \vee \Psi(l')$
2. Ψ preserves the orthocomplementation. That is
$$\Psi(1 - l) = \Psi(1) - \Psi(l)$$
3. Ψ preserves the regular equivalence relation. That is
$$\Psi(l) \sim \Psi(l') \Leftrightarrow l \sim l'$$
   In this case Ψ would automatically preserve the dimension.

We recall that the map $\Psi|_{Min(L)}$ is a bijection between $Min(L)$ and $Min(P_n)$ (by construction).

1. (a) Let $l \leq l'$ and $\Psi(m) \in Min(\Psi(l))$. Then $m \in Min(l) \subseteq Min(l')$ and thus $\Psi(m) \in Min(\Psi(l')$. Consequently, we have $Min(\Psi(l)) \subseteq Min(\Psi(l'))$ and, by lemma 64, $\Psi(l) \leq \Psi(l')$. Conversely, if $\Psi(l) \leq \Psi(l')$, then $Min(\Psi(l)) \subseteq Min(\Psi(l'))$. Thus if $m \in Min(l)$, then $\psi(m) \in Min(\Psi(l))$ and $\psi(m) \in Min(\Psi(l'))$. Consequently $m \in Min(l')$. So we have obtained $Min(l) \subseteq Min(l')$ and then $l \leq l'$ (always using lemma 64).
   (b) Let $\Psi(m) \in Min(P_n)$. We have
$$\Psi(m) \leq \Psi(l \wedge l') \Leftrightarrow m \leq l \wedge l' \Leftrightarrow m \leq l, l' \Leftrightarrow \Psi(m) \leq \Psi(l), \Psi(l') \Leftrightarrow$$
$$\Leftrightarrow \Psi(m) \leq \Psi(l) \wedge \Psi(l')$$

So $\Psi(l \wedge l')$ and $\Psi(l) \wedge \Psi(l')$ have the same minimal elements. Whence they are equals, using lemma 64 again.

(c) Certainly $l, l' \leq l \vee l'$. Thus $\Psi(l), \Psi(l') \leq \Psi(l \vee l')$ and consequently $\Psi(l) \vee \Psi(l') \leq \Psi(l \vee l')$. Conversely, let $\Psi(m) \in Min(\Psi(l \vee l') - \Psi(l) \vee \Psi(l'))$, then

$$\Psi(m) \in Min(\Psi(l \vee l')) \cap \perp (\Psi(l) \vee \Psi(l')) \subseteq$$

using lemma 65

$$\subseteq Min(\Psi(l \vee l')) \cap \perp (\Psi(l)) \cap \perp (\Psi(l'))$$

Whence

$$m \in Min(l \vee l') \cap \perp (l) \cap \perp (l') = \emptyset \quad (by\ 66)$$

That is absurd. Consequenlty $\Psi(l \vee l') - \Psi(l) \vee \Psi(l'))$ does not contain minimal elements and thus it must be zero.

2. One has

$$\Psi(m) \in Min(\Psi(1-l)) \Leftrightarrow m \in Min(1-l) \Leftrightarrow m \in Min(1) \cap \perp (l) \Leftrightarrow$$

$$\Leftrightarrow \Psi(m) \in Min(\Psi(1)) \cap \perp (\Psi(l)) \Leftrightarrow \Psi(m) \in Min(\Psi(1) - \Psi(l))$$

Thus $\Psi(1-l)$ and $\Psi(1) - \Psi(l)$ have the same minimal elements and thus they are equals.

3. It remains to prove that Ψ preserves the regular equivalence relation. Let $\sim_L$ be the equivalence relation on L and let $\sim_{P_n}$ be the equivalence relation on P_n. We have to prove that $l \sim_L l' \Leftrightarrow \Psi(l) \sim_{P_n} \Psi(l')$. Let $l \sim_L l'$, we write $l = \bigvee_{i \in I} l_i$ e $l' = \bigvee_{i \in I} l'_i$, where $\{l_i\}$ and $\{l'_i\}$ are two maximal families of mutually orthogonal and minimal elements dominated respectively by l and l'. Since l_i and l'_i are minimal, it must be $l_i \sim_L l'_i$ and thus, since Ψ preserves minimality, $\Psi(l_i) \sim_{P_n} \Psi(l'_i)$ for each i. Now, since Ψ preserves minimality, orthogonality and order, $\{\Psi(l_i)\}$ and $\{\Psi(l'_i)\}$ are two maximal families of minimal projections of $M_n(\mathbb{C})$ dominated respectively by $\Psi(l)$ and $\Psi(l')$. This means that $\Psi(l)$ and $\Psi(l')$ project onto two subspaces of the same dimension and thus they are equivalent. Conversely, we assume that $\Psi(l)$ and $\Psi(l')$ are equivalent and thus that they project onto subspaces of the same

dimension. We decompose these two subspaces in one-dimensional and mutually orthogonal subspaces and so we write $\Psi(l) = \bigvee_{i\in I} \Psi(l_i)$ and $\Psi(l') = \bigvee_{i\in I} \Psi(l'_i)$, where $\{l_i\}$ and $\{l'_i\}$ are two maximal families of minimal and mutually orthogonal elements dominated respectively by l and l'. Thus $l_i \sim l'_i$ for each i and consequently $l = \bigvee l_i \sim_L \bigvee l'_i = l'$.

□

Remark 71. Some easy consequences of the correspondence theorem:

1. In a factorial W^*-lattice of type I the parallelogram rule is automatically verified.
2. In a factorial W^*-lattice of type I there is only one dimensionable regular equivalence relation, the perspective one.

Remark 72. This remark is suggested by F. Radulescu ([Ra]).
Let L_n be a W^*-lattice of type I_n and P_n the projections lattice of $M_n(\mathbb{C})$. The correspondence theorem guarantees $L_n \cong P_n$. A first consequence is the not obvious inclusion $L_n \hookrightarrow L_{n+1}$. Not obvious because it is not sufficient to chose those elements in L_{n+1} whose dimension is less then $\frac{n}{n+1}$: this is not a lattice! Actually this inclusion does not depend just by the correspondence theorem, but by the existence of references: choose in L_{n+1} a reference $R = \{m_1, ...m_{n+1}\}$, the sublattice generated by $\{m_1, ...m_n\}$ is isomorphic to L_n. This argument shows also that it is possible to choose references on L_n and P_n such that each square of the following diagram commute.

$$\begin{tikzcd}
L_1 \arrow[r, "\xi_1"] \arrow[d] & P_1 \arrow[d] \\
L_2 \arrow[r, "\xi_2"] \arrow[d] & P_2 \arrow[d] \\
\cdots \arrow[d] & \cdots \arrow[d] \\
L_n \arrow[r, "\xi_n"] \arrow[d] & P_n \arrow[d] \\
L_{n+1} \arrow[r, "\xi_{n+1}"] \arrow[d] & P_{n+1} \arrow[d] \\
\cdots & \cdots
\end{tikzcd}$$

ξ_n being the isomorphisms induced by the choice of increasing references. Now, since the hyperfinite factor of type II_1 is the inductive limit of $M_n(\mathbb{C})$ with normalized trace, we can find a factorial W^*-lattice of type II_1 as inductive limit of factorial W^*-lattices of type I_n. Moreover, this inductive limit coincides with the direct limit (with respect the canonical inclusion) completed by cuts (Dedekind-MacNeille completing, see [MacN]).

Problem 73. Does the concept of W^*-lattice of type I axiomatize the projections lattice of a type I finite von Neumann algebra one?

10 A straightening theorem?

In this section we want to describe a possible improvement of the correspondence theorem. Indeed we believe it is possible to remove hypothesis on the existence of an orthogonal complement. This hope can be seem exaggerate and thus vain, but actually there are many cases in which something similar happens. The main one, since it looks like the our case, is the following: a complex linear space V of dimension n is isomorphic to $\mathbb{C}^n$, which has a notion of orthogonality that we can transport on V. On the other hand, if we analyze our construction we notice that the notion of orthogonality is been used only in two step: the first one is to define central elements and then the abelian ones. Nevertheless the notion of factoriality, following von Neumann, does not really depend by the existence of central elements, but it mainly depends by the uniqueness of the inverse (see Def.29); moreover, abelian elements allow only to prove the existence of minimal elements. Thus we can require the existence of a minimal element as an axiom. The second step in which we used the orthogonality is to define references. This is just what happens in the case of a generic linear space of dimension n with respect to $\mathbb{C}^n$: in a similar way, we believe it is possible to start from an affine reference (in the sense of the following definition) and to find a lattice isomorphism with a factorial W^*-lattice of type I_n, n being the cardinality (invariant!) of the affine reference.

Definition 74. An affine reference is a family of minimal elements $R = \{l_i\}$ such that

1. $\bigvee l_i = 1$
2. Each subset of R does not verify the first condition.

Definition 75. An irreducible and continuous lattice with minimal elements is an irreducible complete lattice such that $|L| = |\mathbb{R}|$ and $Min(L) \neq \emptyset$.

Let L be an irreducible and continuous lattice with minimal elements.

Remark 76. The second condition of Def.74 implies that $\{l_i\}$ are completely independents. Indeed, if not, let J and i_0 such that $l_{i_0} \leq \bigvee_{j \in J} l_j$ and $i_0 \notin J$. Then $\bigvee_{i \in I \setminus \{i_0\}} l_i$ is still equal to 1. Consequently (in the case of irreducible and complete lattices) we can still apply th.44 and obtain that two affine references have the same finite cardinality.

Proposition 77. *Each element of L is sup of a family of minimal elements.*

Proof. Applying Prop. 37 (choosing l minimal instead abelian!) we have that each element $l \in L$ contains minimal elements. If $\bigvee_{l' \in Min(l)} l' < l$ we apply a classical lemma of J. von Neumann: if $l' \leq l$, there exists l'' (independent to l'!) such that $l' \vee l'' = l$. Take a minimal element contained in l' and find the absurd. □

Conjecture 78. (Straightening theorem?) Let L be an irreducible and continuous lattice with minimal elements and let n be the cardinality of one of its affine reference. There exists an involutory anti-automorphism $\perp: L \to L$ such that $(L, \leq, \perp)$ is a factorial lattice of type I_n.

Remark 79. This should be the best theorem we can find, because it is quite simple to prove that the axioms of irreducible and continuous lattice with minimal elements are independent: we can find examples of lattices satisfying all of them except one.

11 The separable case

The main obstacle that we can find in order to extend the correspondence theorem to the separable case is the absence of the modular property. But it is been used only to prove that if $l' \leq l$, one can define $l - l'$. And this last property is been used only to prove that if there exists a minimal element, then each element is sup of a family of minimal elements. We think that there are no other way that assume this property as an axiom.

Definition 80. A factorial W^*-lattice of type I_ω is a factorial W^*-lattice in which

1. Each element is sup of a family of minimal elements.
2. 1 is sup of a countable family of minimal elements.

Theorem 81. *There exists only one (up to lattice isomorphism) factorial W^*-lattice of type I_ω and it is the projection lattice of $B(H)$, where H is a separable Hilbert space.*

Proof. The proof is the same as in the finite dimensional case. Indeed results in Sect.8 still hold and we can still make the construction of the proof of the correspondence theorem thanks to the two properties 1. and 2. of Def.80. $\square$

We can also conjectured the straightening theorem in the separable case.

Conjecture 82. Let L be an irreducible, complete and continuous lattice such that

1. Each element is sup of a family of minimal elements.
2. 1 is sup of a countable family of minimal elements.

Then there exists an involutory anti-automorphism $\perp: L \to L$ such that $(L, \leq, \perp)$ is a factorial lattice of type I_ω.

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